

THE REAL POWERS OF THE CONVOLUTION OF A GAMMA DISTRIBUTION AND A BERNOULLI DISTRIBUTION

Ben Salah Nahla, Masmoudi Afif

*Laboratory of Probability and Statistics, Faculty of Science of Sfax, Sfax University. B. P. 1171
Sfax. Tunisia*

Abstract

In this paper, we essentially compute the set of $x, y > 0$ such that the mapping $z \mapsto \left(1 - r + re^z\right)^x \left(\frac{\lambda}{\lambda - z}\right)^y$ is a Laplace transform. If X and Y are two independent random variables which have respectively Bernoulli and Gamma distributions, we denote by μ the distribution of $X + Y$. The above problem is equivalent to finding the set of $x > 0$ such that μ^{*x} exists.

Key words: BERNOULLI LAW, CONVOLUTION POWER, GAMMA DISTRIBUTION, JØRGENSEN SET, LAPLACE TRANSFORM.

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1. Introduction and Preliminaries

We introduce first some notations and review some basic concepts concerning the Laplace transform and Jørgensen set. For more details, we refer the reader to [1, 3]. For a positive measure μ on \mathbb{R} , we denote by

$$L_\mu : \mathbb{R} \longrightarrow (0; +\infty); \theta \longmapsto \int_{\mathbb{R}} e^{\theta x} \mu(dx)$$

its Laplace transform, and let

$$\Theta(\mu) = \text{interior}\{\theta; L_\mu(\theta) < +\infty\}.$$

We denote by $\mathcal{M}(\mathbb{R})$ the set of such measures on \mathbb{R} such that, furthermore, μ is not concentrated on a point and such that $\Theta(\mu)$ is not empty.

*Corresponding author

**Principal corresponding author

Email addresses: bensalahnahla@yahoo.f (Ben Salah Nahla), Afif.Masmoudi@fss.rnu.tn (Masmoudi Afif)

For $\mu \in \mathcal{M}(\mathbb{R})$, L_μ is real analytic and strictly convex on $\Theta(\mu)$.

The Jørgensen parameter is the parameter corresponding to the power of convolution such that it is the variance in the Gaussian distribution and it is the shape parameter in the Gamma distribution.

Let μ be an element of $\mathcal{M}(\mathbb{R})$, the Jørgensen set $\Lambda(\mu)$ of μ is the set of $x > 0$ such that there exists $\mu_x \in \mathcal{M}(\mathbb{R})$ with $\Theta(\mu_x) = \Theta(\mu)$ and $L_{\mu_x}(\theta) = (L_\mu(\theta))^x$ (see [3]).

In this case, μ_x is called the x^{th} convolution power of μ .

If x and x' are two elements of the Jørgensen set $\Lambda(\mu)$ of μ , then $(\mu_x, \mu_{x'})$ is convolvable since $\Theta(\mu_x) = \Theta(\mu_{x'})$. Furthermore $x + x' \in \Lambda(\mu)$. Hence, $\Lambda(\mu)$ is a semigroup under addition and

$$\mu_{x+x'} = \mu_x * \mu_{x'}.$$

It contains 1 by definition, therefore it contains the set \mathbb{N}^* of non-negative integers. The calculation of $\Lambda(\mu)$ is sometimes a hard problem: it is \mathbb{N}^* when μ is the Bernoulli distribution on $\{0, 1\}$.

A probability distribution μ on \mathbb{R} is infinitely divisible if, for every integer n , there exists a distribution μ_n such that

$$\mu = \mu_n^{*n}$$

that is, μ is the n^{th} power of convolution of μ_n .

In other words, μ is infinitely divisible if, for each integer n , it can be represented as the distribution of the sum $S_n = X_{1,n} + X_{2,n} + \dots + X_{n,n}$ of n independent random variables with a common distribution μ_n (see [2]).

If μ is an element of $\mathcal{M}(\mathbb{R})$, then μ is infinitely divisible if and only if its Jørgensen set $\Lambda(\mu)$ is equal to $(0, +\infty)$.

Let ν be a distribution on the real line having Laplace transform and which is not infinitely divisible. Consider now an infinitely divisible distribution ν' on the real line, also having Laplace transform. We denote by ν'_y the distribution such that $L_{\nu'_y} = L_{\nu'}^y$.

Let $\mu = \nu * \nu'$ be the convolution product of ν and ν' .

Letac et al. [4] considered the case where ν is Bernoulli and ν' is negative Binomial. In this case, the problem is equivalent to finding the set of $(x, y) \in (0, +\infty)^2$ such that there exists a probability $\mu_{x,y}$ on the real line with Laplace transform $L_{\nu}^x L_{\nu'}^y$.

In this work, we consider the case where ν is Bernoulli and ν' is Gamma. In this situation, the techniques are completely different since we use essentially the analyticity of the Laplace transform.

For fixed $r \in (0, 1)$ and $\lambda > 0$, the present paper wonders for which values of $x, y > 0$ the function defined on $(-\infty, \lambda)$ by $z \mapsto \left(1 - r + re^z\right)^x \left(\frac{\lambda}{\lambda - z}\right)^y$ is the Laplace transform of a probability. This function is the Laplace transform of the function

$$t \mapsto \frac{\lambda^y (1-r)^x}{\Gamma(y)} \sum_{k \geq 0} \frac{x(x-1)(x-2)\dots(x-k+1)}{k!} \left(\frac{r}{1-r}\right)^k e^{-\lambda(t-k)} (t-k)_+^{y-1}$$

where a_+ means $\max(a, 0)$. Denote $R = \frac{r}{1-r}e^\lambda$ for simplicity. The problem is therefore equivalent in finding the set Λ_R of the (x, y) 's such that

$$t \mapsto \frac{\lambda^y(1-r)^x}{\Gamma(y)} \sum_{k \geq 0} \frac{x(x-1)(x-2)\dots(x-k+1)}{k!} R^k e^{-\lambda t} (t-k)_+^{y-1} = f(t)$$

is a positive function for all t (note that this series converges, having a general term $\rightarrow 0$ for k large enough when $t > 0$ is fixed).

The present paper determines Λ_R in Section 2.

2. Result

Let X and Y be two independent random variables following the Bernoulli $\mathcal{B}(r)$ distribution with expectation r and the Gamma $\gamma(a, \lambda)$ distribution respectively

$$P(X = 0) = 1 - r, \quad P(X = 1) = r \in (0, 1),$$

and

$$\gamma(a, \lambda)(dt) = \frac{\lambda^a}{\Gamma(a)} t^{a-1} e^{-\lambda t} \mathbf{1}_{(0, +\infty)}(t) dt.$$

The law of $X + Y$ can be seen as the mixture of Gamma distributions up to translation that is

$$\begin{aligned} \mu &= \mathcal{B}(r) * \gamma(a, \lambda) \\ &= (1-r)\gamma(a, \lambda) + r\gamma(a, \lambda) * \delta_1, \end{aligned}$$

where δ_1 denotes the Dirac measure at 1.

Now, we state our main result. The following statement determines the set of $x > 0$ such that μ^{*x} exists, or equivalently, in finding the set Λ_R of (x, y) 's such that

$$t \mapsto \frac{\lambda^y(1-r)^x}{\Gamma(y)} \sum_{k \geq 0} \frac{x(x-1)(x-2)\dots(x-k+1)}{k!} R^k e^{-\lambda t} (t-k)_+^{y-1} = f(t)$$

is a positive function for all t .

Theorem 2.1. *a) If $R \leq 1$, then $\Lambda_R = (0, +\infty) \times [1, +\infty)$.*

b) If $R > 1$, then $\Lambda_R = \mathbb{N} \times (0, +\infty)$.

PROOF. For $R \leq 1$: for fixed $x > 0$, we define the positive integer $k_0 = k_0(x)$ by $k_0 - 1 \leq x < k_0$ and

$$f_n(t) = \frac{\lambda^y(1-r)^x}{\Gamma(y)} \sum_{k=0}^n \frac{x(x-1)(x-2)\dots(x-k+1)}{k!} R^k e^{-\lambda t} (t-k)_+^{y-1}$$

and note that $f_n(t) > 0$ for all $t > 0$ and for all $n \leq k_0$. For seeing $\Lambda_R \subset (0, +\infty) \times [1, +\infty)$ assume $(x, y) \in \Lambda_R$ but $y < 1$ and consider f in the interval $(k_0 + 1, k_0 + 2)$

$$f(t) = \frac{\lambda^y(1-r)^x}{\Gamma(y)}(x-k_0)\frac{x(x-1)(x-2)\dots(x-k_0+1)}{(k_0+1)!}R^{k_0+1}e^{-\lambda t}(t-k_0-1)^{y-1} + f_{k_0}(t).$$

Since $f_{k_0}(t) > 0$ and $(x-k_0) < 0$ we have the contradiction $\lim_{t \rightarrow k_0+1} f(t) = -\infty$. To show that $\Lambda_R \supset (0, +\infty) \times [1, +\infty)$, we fix $x > 0$ and $y \geq 1$ and we show that $f(t) \geq 0$ for all t . This is already true for $t < k_0 + 1$ since $f(t) = f_{k_0}(t)$ in that case. For $k_0 + 1 \leq k_1 \leq t < k_1 + 1$ where k_1 is an integer, we use the alternate series trick: consider the positive finite sequence $(u_k)_{k=k_0}^{k_1}$ defined by

$$u_k = (-1)^{k-k_0} \frac{x(x-1)(x-2)\dots(x-k+1)}{k!} R^k (t-k)^{y-1}$$

which is decreasing since for $k_0 \leq k < k_1$ we have

$$\frac{u_{k+1}}{u_k} = R \times \frac{k-x}{k+1} \left(\frac{t-k-1}{t-k} \right)^{y-1} < 1$$

and thus

$$f(t) = f_{k_0-1}(t) + \frac{\lambda^y(1-r)^x}{\Gamma(y)} \sum_{k=k_0}^{k_1} (-1)^{k-k_0} u_k \geq 0.$$

ii) For $R > 1$: of course $\Lambda_R \supset \mathbb{N} \times (0, +\infty)$ is trivial. To prove that $\Lambda_R \subset \mathbb{N} \times (0, +\infty)$, suppose that there exists $(x, y) \in \Lambda_R$ such that x is not an integer. Since

$z \mapsto \left(1 + \frac{r}{1-r}e^z\right)^x \left(\frac{\lambda}{\lambda-z}\right)^y$ is real analytic on $(-\infty, \lambda)$ it is analytic on the strip $S = (-\infty, \lambda) + i\mathbb{R}$. However, since $R > 1$, or $\log(\frac{1-r}{r}) < \lambda$ this implies that $z \mapsto 1 + \frac{r}{1-r}e^z$ has a zero in the strip S namely $z_0 = i\pi + \log(\frac{1-r}{r})$. But the fact that x is not an integer prevents $z \mapsto \left(1 + \frac{r}{1-r}e^z\right)^x$ from being analytic on z_0 and we get the desired contradiction.

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References

- [1] Barndorff-Nielsen, O.: *Information and Exponential Families in Statistical Theory*, Wiley, New York (1978)
- [2] Feller, W.: *An Introduction to Probability Theory and Its Applications*, Wiley, New York (1971)

- [3] Jørgensen, B.: *The Theory of Dispersion Models*, London, Chapman & Hall (1997)
- [4] Letac, G., Malouche, D., Maurer, S.: The real powers of the convolution of a negative Binomial distribution and a Bernoulli distribution. *Proc. A.M.S.* **130**, 2107-2114 (2002)